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Fermionic relatives of Stirling and Lah numbers

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Abstract
In this paper certain 'fermionic' Stirling numbers introduced recently are discussed. Roughly speaking, these numbers are obtained by taking the 'fermionic' limit $q \to -1$ of the q-deformed Stirling numbers. The usual Stirling numbers correspond in this language to the 'bosonic' limit $q \to 1$. It is shown that the fermionic Stirling numbers are given by binomial coefficients and that they satisfy the same relations as the undeformed Stirling numbers. The fermionic relatives of Lah numbers are also very briefly discussed.

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1. Introduction

The Stirling numbers of second kind $S(n,k)$ play an important role in many combinatorial problems [1, 2] and have also appeared in the physical context of normal ordering bosonic creation and annihilation operators [3]. For $q \in [0,1]$ certain q-deformed bosonic operators may be introduced [4] which generalize the undeformed bosonic ones (corresponding to $q = 1$). It was shown in [5] that the corresponding normal ordering problem leads to the q-deformed Stirling numbers $S_q(n,k)$ in the version introduced by Milne [6]. These q-deformed Stirling numbers were introduced by Carlitz [7, 8] (in a slightly different form) and have been discussed (in slightly varying forms) in various contexts, see, e.g., [6–14]. In the limit $q \to 1$ one obtains the usual 'bosonic' Stirling numbers, i.e., $S_{q=1}(n,k) = S(n,k)$. It is also possible to consider the case $q \in (-1,0)$ corresponding to q-deformed fermionic creation and annihilation operators, see [15] and the literature given therein. The expression (and the recursion relation) for the $S_q(n,k)$ remains the same, so it is tempting to consider the limit $q \to -1$ as suggested in [15]. This limit will be called 'fermionic' since the corresponding creation and annihilation operators are those of an undeformed fermion. Since taking the limit $q \to -1$ for $S_q(n,k)$ is not straightforward (if we use the standard expression for them given below), we define the fermionic Stirling numbers $S_f(n,k)$ as solutions of the corresponding recursion relation. Due to the unconventional form of this recursion relation, we will determine $S_f(n,k)$ in a straightforward fashion. Surprisingly, they are given by a single binomial coefficient (recall that the usual Stirling numbers $S(n,k)$ can be expressed as a sum of n binomial coefficients). In the same fashion fermionic Stirling numbers of first kind $s_f(n,k)$ are introduced and discussed. It is shown that $s_f(n,k)$ is also given by a single binomial coefficient. Furthermore, it is shown that the inversion relation combining the Stirling numbers of both kinds as well as the interpretation of the Stirling numbers as connection coefficients holds true also in this context. A similar treatment of the Lah numbers shows that their fermionic relative is rather uninteresting. In a very recent preprint [16], some of these questions are also discussed from a different perspective.

2. The q-deformed Stirling numbers

Let $q \in (-1, 1]$. Following [17] we introduce the standard notation

$$[n] \equiv [n]_q = (1 + q + \cdots + q^{n-1}) = \frac{1-q^n}{1-q}$$

for q-deformed numbers and

$$[n]! \equiv [n][n-1]\cdots[2][1] \qquad [n;k] \equiv \frac{[n]!}{[n-k]![k]!} \tag{1}$$

for the q-factorials and q-binomial coefficients. Note that we will suppress the index q in the following as far as possible, only displaying it when necessary. In the limit $q \to 1$ one obtains $[n]_{q=1} = n$. Let us point out that in the case $q < 0$ we may write $q \equiv -\tilde{q}$ with a positive $\tilde{q}$. It follows that

$$[n]_q \equiv \frac{1-q^n}{1-q} = \frac{1-(-\tilde{q})^n}{1+\tilde{q}} \equiv [n]^F_{\tilde{q}}$$

here the last equation is the definition of the $\tilde{q}$-fermionic basic number appearing in recent studies of the $\tilde{q}$-deformed fermionic oscillator [18–21]. Note that the limit $q \to -1$, i.e., $\tilde{q} \to 1$, yields

$$[n]_{q=-1} = [n]^F_{\tilde{q}=1} = \frac{1-(-1)^n}{2} = \epsilon_n := \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ 1 & \text{if} \quad n \text{ is odd.} \end{cases} \tag{2}$$

In particular, this implies for $n \geqslant 2$ that

$$[n]_{q=-1}! = 0. \tag{3}$$

The q-deformed Stirling numbers of second kind $S_q(n,k)$ are given by (in the version of Milne [6])

$$S_q(n,k) = \sum_{p=1}^{k} (-1)^{k-p} q^{\binom{k-p}{2}} \frac{[p]_q^{n-1}}{[p-1]_q![k-p]_q!} \tag{4}$$

where $n, k \in \mathbf{N}$ with $k \leqslant n$. They satisfy the recursion relation

$$S_q(n+1,k) = q^{k-1} S_q(n,k-1) + [k]_q S_q(n,k) \tag{5}$$

with initial values $S_q(1,0) = 0$ and $S_q(1,1) = 1$ (in the following we will also use the convention $S_q(n,0) := 0$). The recursion relation implies that the $S_q(n,k)$ are polynomials in q (although this is not obvious from (4)). In the 'bosonic' limit $q \to 1$ one obtains the usual Stirling numbers, i.e., $S_{q=1}(n,k) = S(n,k)$, and their usual recursion relation [1, 2]. Some simple examples for the q-deformed case are given by $S_q(n,1) = 1$, $S_q(n,2) = [2]_q^{n-1} - 1$, $S_q(n,n) = q^{\frac{n(n-1)}{2}}$. Note that due to (3) it is not straightforward to take the limit $q \to -1$ in (4) (but recall that the $S_q(n,k)$ are in fact polynomials in q; an explicit expression can be found in [14] and is the starting point for [16]). The q-deformed Bell numbers are defined by $B_q(n) = \sum_k S_q(n,k)$. Milne showed [6] that there exists a q-deformed Dobinski relation

$$B_q(n) = \frac{1}{e_q(1)} \sum_{k=1}^{\infty} \frac{[k]^n}{[k]!} \tag{6}$$

involving the q-exponential function $e_q(x) = \sum_{m=0}^{\infty} \frac{x^m}{[m]!}$. The q-deformed Stirling numbers of first kind $s_q(n,k)$ satisfy the recursion relation

$$s_q(n+1,k) = q^{-n}\{s_q(n,k-1) - [n]_q s_q(n,k)\} \tag{7}$$

with initial values $s_q(1,0) = 0$ and $s_q(1,1) = 1$. Denoting for $k \geqslant 1$ the falling factorials by $[x]_q^{\underline{k}} := [x]_q[x-1]_q \cdots [x-k+1]_q$, the q-deformed Stirling numbers may also be defined as connection coefficients, i.e.,

$$[x]_q^n = \sum_{k=0}^{n} S_q(n,k)[x]_q^{\underline{k}} \qquad [x]_q^{\underline{n}} = \sum_{k=0}^{n} s_q(n,k)[x]_q^k. \tag{8}$$

From this it is easy to show that the q-deformed Stirling numbers satisfy for $n \geqslant m$ the inversion relations (reproducing those of the ordinary Stirling numbers in the limit $q \to 1$)

$$\sum_{k=m}^{n} s_q(n,k) S_q(k,m) = \delta_{nm} \qquad \sum_{k=m}^{n} S_q(n,k) s_q(k,m) = \delta_{nm}. \tag{9}$$

3. The fermionic Stirling numbers

The *fermionic Stirling numbers of first kind* are defined as solutions of the recursion relation

$$s_f(n+1,k) = (-1)^n s_f(n,k-1) + (-1)^{n+1} \epsilon_n s_f(n,k) \tag{10}$$

with initial values $s_f(1,0) = 0$ and $s_f(1,1) = 1$ (we will also use the convention $s_f(n,0) := 0$). The *fermionic Stirling numbers of second kind* are defined as solutions of the recursion relation

$$S_f(n+1,k) = (-1)^{k-1} S_f(n,k-1) + \epsilon_k S_f(n,k) \tag{11}$$

with initial values $S_f(1,0) = 0$ and $S_f(1,1) = 1$ (we will also use the convention $S_f(n,0) := 0$). Note that (10) is the 'fermionic' limit $q \to -1$ of (7) and that (11) is obtained from (5) in the 'fermionic' limit $q \to -1$. Due to the unusual form of the recursion relation, the standard approach via generating functions [22] does not seem applicable straightforwardly (in contrast to the case $q \in (-1,1]$ where it works directly [15]). We will instead examine the recursion relations directly. As a first step we determine the values $S_f(n,k)$ for maximal and small k. They are given by $S_f(n,n) = (-1)^{\frac{n(n-1)}{2}}$ as well as

$$S_f(n,1) = 1 \qquad S_f(n,2) = -1 \qquad S_f(n,3) = 2-n \qquad S_f(n,4) = n-3.$$

Note that these results coincide with those obtained by taking the limit $q \to -1$ for the expressions given above. These results may be shown directly using (11), but follow also from the general formula (15). Let us turn to the Stirling numbers of first kind. It follows immediately from (10) that $s_f(n,n) = (-1)^{\frac{n(n-1)}{2}}$. Due to the factor '$\epsilon_n$' in (10) many Stirling numbers of first kind vanish. More precisely, one has

$$s_f(n,k) = 0 \qquad \text{for } n > 2k.$$

We show this by induction over k. Let us assume that $s_f(n,\tilde{k}) = 0$ for $1 \leqslant \tilde{k} \leqslant k$ and $n > 2\tilde{k}$. We now want to show that $s_f(n, k+1) = 0$ for $n \geqslant 2k+3$. From (10) it follows that $s_f(2k+3, k+1) = s_f(2k+2, k) - \epsilon_{2k+2} s_f(2k+2, k+1)$. The first summand vanishes due to the induction hypothesis, whereas the second summand vanishes due to $\epsilon_{2k+2} = 0$. Thus, $s_f(2k+3, k+1) = 0$. This implies (via the recursion relation and the induction hypothesis) that all $s_f(n,k)$ with $n \geqslant 2k+3$ vanish, as we wanted to show. Note that this implies that for a given n roughly the first $\frac{n}{2}$ Stirling numbers $s_f(n,k)$ vanish (a more precise statement is given below after (17)). In the following we will also use the notation

$$[n]_f := [n]_{q=-1} = \epsilon_n$$

for the fermionic basic numbers, see (2). They satisfy $[n+m]_f = [m]_f + (-1)^m [n]_f$, in particular $[n+1]_f = 1 - [n]_f$. Although we have not defined the fermionic Stirling numbers as limit $q \to -1$ of the q-deformed Stirling numbers, the analogue of (8) holds true. Thus, the fermionic Stirling numbers are connection coefficients for the fermionic basic numbers, i.e.,

$$[x]_f^n = \sum_{k=0}^{n} S_f(n,k)[x]_f^{\underline{k}} \qquad [x]_f^{\underline{n}} = \sum_{k=0}^{n} s_f(n,k)[x]_f^k. \tag{12}$$

Let us prove the first equation by induction over n. Let us assume that the assertion holds up to n. Using the induction hypothesis implies that $[x]_f^{n+1} = \sum_{k=0}^{n} S_f(n,k)[x]_f[x]_f^{\underline{k}}$. Now, we use $[x]_f = [k]_f + (-1)^k[x-k]_f$ to obtain

$$[x]_f^{n+1} = \sum_{k=0}^{n} \{S_f(n,k)[k]_f + (-1)^k[x-k]_f S_f(n,k)\}[x]_f^{\underline{k}}.$$

Recalling $[k]_f \equiv \epsilon_k$ and the recursion relation (11) yields $[x]_f^{n+1} = \sum_{k=0}^{n+1} S_f(n+1,k)[x]_f^{\underline{k}}$. Thus, the assertion is proved. The second equation is proved in a similar fashion. Note that we have used in the proof the recursion relations of the fermionic Stirling numbers to show that they satisfy (12). In the standard approach the q-deformed Stirling numbers are defined as connection coefficients (8) and an argument similar to that above is used to derive the recursion relations, see [6]. In contrast to the usual (or q-deformed) situation, the content of (12) is rather meagre. Recall that $[x]_f^{\underline{k}}$ vanishes provided that $k \geqslant 2$ (since either x or $x-1$ is even), so that from the first equation of (12) only $[x]_f^n = S_f(n,1)[x]_f = [x]_f$ remains. Of course, this is an aspect of the fermionic nature of the situation. Since we did not define the fermionic Stirling numbers directly as the limits $q \to -1$ of the q-deformed Stirling numbers, it is at first not clear whether the analogue of (9) holds true. Nevertheless, it is easy to show that the fermionic Stirling numbers satisfy the inversion relations

$$\sum_{k=m}^{n} s_f(n,k)S_f(k,m) = \delta_{nm} \qquad \sum_{k=m}^{n} S_f(n,k)s_f(k,m) = \delta_{nm}. \tag{13}$$

Let us consider the second equation. Inserting the second equation of (12) into the first equation of (12) yields

$$[x]_f^n = \sum_{m=0}^{n} \left\{ \sum_{k=m}^{n} S_f(n,k)s_f(k,m) \right\} [x]_f^{\underline{m}}$$

from which the assertion follows. The first equation is shown similarly. Although the next observation follows trivially from the recursion relation, it is crucial for the following to determine the explicit values of the fermionic Stirling numbers of second kind. In the case that k is even, (11) reduces to $S_f(n+1,k) = -S_f(n,k-1)$; in the case that k is odd, it reduces to $S_f(n+1,k) = S_f(n,k) + S_f(n,k-1)$. This shows that the case of even k may be reduced to the case of odd k in a trivial manner. Thus, it is enough to determine $S_f(n,k)$ for odd k. For odd $k \geqslant 3$ one finds that

$$S_f(n+1,k) = S_f(n,k) - S_f(n-1,k-2). \tag{14}$$

This recursion relation involves only odd k. Let us introduce new sequences $T(n,l)$ by $T(n,l) := S_f(n,2l+1)$. Due to the 'scaling' of the second argument, the new sequences satisfy the simpler recursion relation $T(n+1,l) = T(n,l) - T(n-1,l-1)$. This may be solved either with the help of generating functions [22] or by direct inspection, yielding $T(n,l) = (-1)^l\binom{n-l-1}{l}$ and, therefore, $S_f(n,2l+1) = (-1)^l\binom{n-l-1}{l}$. Let us denote by $\lfloor x\rfloor$ the greatest integer less than or equal to x. In general, the fermionic Stirling numbers of second kind are given by

$$S_f(n,k) = (-1)^{\lfloor \frac{k}{2}\rfloor}\binom{n-\lfloor\frac{k}{2}\rfloor-1}{\lfloor\frac{k-1}{2}\rfloor}. \tag{15}$$

Let us first consider the case of odd k. In this case we may write $k = 2l+1$ with $l = \lfloor\frac{k}{2}\rfloor$. This reproduces the result given above. Let us now consider the case of even k. In this case we may write $k = 2\lfloor\frac{k}{2}\rfloor$ to obtain $S_f(n,k) = -S_f\big(n-1, 2\big\{\lfloor\frac{k}{2}\rfloor - 1\big\} + 1\big)$. Using now the formula for odd k proves the asserted formula (15). The same expression (15) has been found in [16]. Having determined the fermionic Stirling numbers, it would be nice to have a convenient expression for the corresponding *fermionic Bell numbers* $B_f(n) := \sum_{k=1}^{n} S_f(n,k)$. This has been determined in [16] and can be stated as follows:

$$B_f(n) = \begin{cases} (-1)^n & \text{if} \quad n \equiv 0 \pmod 3 \\ (-1)^{n+1} & \text{if} \quad n \equiv 1 \pmod 3 \\ 0 & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

Furthermore, the exponential generating function for the fermionic Bell numbers is also given in [16]. It is clear that taking the limit $q \to -1$ in (6) to obtain a *fermionic Dobinski relation* is not straightforward. Let us, therefore, turn to the Stirling numbers of first kind. Similar to the case of the Stirling numbers of second kind, we consider the case of even or odd n. In the case that n is even, (10) reduces to $s_f(n+1,k) = s_f(n,k-1)$; in the case that n is odd, it reduces to $s_f(n+1,k) = s_f(n,k) - s_f(n,k-1)$. This shows that the case of odd n may be reduced to the case of even n in a trivial manner. Thus, it is enough to determine the $s_f(n,k)$ for even n. One obtains for $n = 2l$ that

$$s_f(2l+2,k) = s_f(2l,k-1) - s_f(2l,k-2). \tag{16}$$

Note that this contains only even numbers as first argument. The introduction of $R(l,k) := s_f(2l,k)$ (where $0 \leqslant k \leqslant 2l$) yields the simpler recursion relation $R(l+1,k) = R(l,k-1) - R(l,k-2)$ with solution $R(l,k) = (-1)^{k-l}\binom{l}{k-l}$; note that this is valid also for $0 \leqslant k \leqslant l-1$, yielding $R(l,k) = 0$. This shows that $s_f(2l,k) = (-1)^{k-l}\binom{l}{k-l}$ for $0 \leqslant k \leqslant 2l$. In general, the fermionic Stirling numbers of first kind are given by

$$s_f(n,k) = (-1)^{k-\lfloor\frac{n+1}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor}{k-\lfloor\frac{n+1}{2}\rfloor}. \tag{17}$$

In particular, $s_f(n,k) = 0$ for $k < \lfloor\frac{n+1}{2}\rfloor$. Let us first consider the case of even n. In this case we may write $n = 2\lfloor\frac{n}{2}\rfloor$ which reproduces the result given above. Let us now consider the case of odd n. In this case we may write $n = 2\lfloor\frac{n}{2}\rfloor + 1$ to obtain $s_f(n,k) = s_f\left(2\lfloor\frac{n}{2}\rfloor, k-1\right)$. Using now the formula for even n proves the assertion.

4. The fermionic Lah numbers

It was shown in [23, 24] that in the same physical context of normal ordering bosonic operators where the Stirling numbers appear, the (unsigned) Lah numbers $L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}$ also appear. It was shown in [15] that in the corresponding q-deformed situation, the q-deformed Lah numbers $L_q(n,k)$ introduced in [10, 11] appear (note that in [14] a different version of q-deformed Lah numbers is defined). These q-deformed Lah numbers are given by

$$L_q(n,k) = q^{k(k-1)}\frac{[n]_q!}{[k]_q!}[n-1;k-1]_q \tag{18}$$

and satisfy the recursion relation

$$L_q(n+1,k) = q^{n+k-1}L_q(n,k-1) + [n+k]_q L_q(n,k) \tag{19}$$

with initial values $L_q(1,0) = 0$ and $L_q(1,1) = 1$ [15]. Denoting the rising factorials by $[x]_q^{\overline{n}} := [x]_q[x+1]_q\cdots[x+n-1]_q$, the q-deformed Lah numbers may also be defined as connection coefficients, i.e.,

$$[x]_q^{\overline{n}} = \sum_{k=0}^{n} L_q(n,k)[x]_q^{\underline{k}}. \tag{20}$$

Taking the 'bosonic' limit $q \to 1$ reproduces the usual Lah numbers, i.e., $L_{q=1}(n,k) = L(n,k)$. Taking the 'fermionic' limit $q \to -1$ is again not straightforward, so we will instead define the *fermionic Lah numbers* $L_f(n,k)$ as solutions of the recursion relation

$$L_f(n+1,k) = (-1)^{n+k-1}L_f(n,k-1) + \epsilon_{n+k}L_f(n,k) \tag{21}$$

with initial values $L_f(1,0) = 0$ and $L_f(1,1) = 1$. Note that (21) is the 'fermionic' limit $q \to -1$ of (19). Although we have not defined the fermionic Lah numbers as limit $q \to -1$ of the q-deformed Lah numbers, the equation resulting from (20) by considering $q \to -1$ is nonetheless true. Thus, the fermionic Lah numbers are the connection coefficients between rising and falling factorials of fermionic basic numbers, i.e.,

$$[x]_f^{\overline{n}} = \sum_{k=0}^{n} L_f(n,k)[x]_f^{\underline{k}}. \tag{22}$$

Let us prove this by induction over n. Assume that the assertion holds up to n. Since $[x]_f^{\overline{n+1}} = [x+n]_f[x]_f^{\overline{n}}$, use of the induction hypothesis implies that $[x]_f^{\overline{n+1}} = \sum_{k=0}^{n} L_f(n,k)[x+n]_f[x]_f^{\underline{k}}$. Since $[x+n]_f = [n+k]_f + (-1)^{n+k}[x-k]_f$, this equals

$$[x]_f^{\overline{n+1}} = \sum_{k=0}^{n}\{L_f(n,k)[n+k]_f + (-1)^{n+k}[x-k]_f L_f(n,k)\}[x]_f^{\underline{k}}.$$

Recalling $[n+k]_f \equiv \epsilon_{n+k}$ and using the recursion relation (21) yields $[x]_f^{\overline{n+1}} = \sum_{k=0}^{n+1} L_f(n+1,k)[x]_f^{\underline{k}}$, proving the assertion. Unfortunately, a closer look reveals that the fermionic Lah numbers are rather uninteresting since

$$L_f(n,k) = \delta_{n,k}. \tag{23}$$

Let us prove this by induction. The assertion is trivially true for $n = 1$ since the asserted values are the prescribed initial values. Let us assume that the assertion holds up to n. For $n+1$ we use the defining relation (21) to express $L_f(n+1,k)$ as a sum of two terms of the form $L_f(n,*)$. For $0 \leqslant k \leqslant n-1$ both terms on the right-hand side vanish due to the induction hypothesis, thus $L_f(n+1,k) = 0$ for $0 \leqslant k \leqslant n-1$. In the case $k = n$ the first term on the right-hand side vanishes due to the induction hypothesis, whereas the second term vanishes since $\epsilon_{2n} = 0$. Thus, $L_f(n+1, n) = 0$. In the remaining case $k = n + 1$, the second term on the right-hand side vanishes, whereas the first one gives $(-1)^{2n} L_f(n, n) = 1$ due to the induction hypothesis. This shows (23). Therefore, the exponential generating function of the fermionic Lah numbers is given by

$$L_f(x; k) := \sum_{n\geqslant 0} L_f(n, k)\frac{x^n}{n!} = \frac{x^k}{k!}.$$

It is interesting to compare this to the exponential generating function of the ordinary Lah numbers $L(n, k)$ given by

$$L(x; k) := \sum_{n\geqslant 0} L(n, k)\frac{x^n}{n!} = \frac{1}{k!}\left(\frac{x}{1-x}\right)^k = \frac{x^k}{k!}(1-x)^{-k}.$$

5. Discussion

In this paper we have considered the fermionic Stirling and Lah numbers as solutions of certain recursion relations which are obtained in the limit $q \to -1$ from those of the q-deformed case. Explicit expressions are given in (15), (17) and (23). Since the q-deformed case corresponds in the associated physical situation to an interpolation between a bosonic ($q = 1$) and a fermionic (in the limit $q \to -1$) system, the limit $q \to -1$ was called 'fermionic' [15]. The commutation relations of the q-deformed fermionic oscillator introduced in [18, 19] may be written in the form

$$b_q^\dagger b_q - q b_q b_q^\dagger = 1 \qquad \left[N, b_q^\dagger\right] = b_q^\dagger \qquad [N, b_q] = -b_q \tag{24}$$

where N is the q-deformed number operator (see also [15]). Note that these relations reduce in the limit $q \to -1$ to those of the usual fermionic oscillator, except that for $b \equiv b_{q=-1}$ and $b^\dagger \equiv b_{q=-1}^\dagger$ the relations $b^2 = 0 = (b^\dagger)^2$ do *not* hold. This means that the exclusion principle does not hold in the limit $q \to -1$. For $q > -1$ one has the relation $\left(b_q^\dagger b_q\right)^n = \sum_{k=1}^n S_q(n, k)\left(b_q^\dagger\right)^k b_q^k$, yielding in the limit $q \to -1$

$$(b^\dagger b)^n = \sum_{k=1}^{n} S_f(n, k)(b^\dagger)^k b^k. \tag{25}$$

The creation and annihilation operators $\tilde{b}^\dagger$ and $\tilde{b}$ of the usual fermionic oscillator satisfy, in addition to the relations satisfied by $b^\dagger$ and b, the relations $(\tilde{b}^\dagger)^2 = 0 = b^2$. This implies that in the corresponding equation $(\tilde{b}^\dagger\tilde{b})^n = \sum_{k=1}^n \tilde{S}_f(n, k)\left(\tilde{b}^\dagger\right)^k \tilde{b}^k$ the coefficients $\tilde{S}_f(n, k)$ are arbitrary for $k > 1$ and may be chosen for convenience as $\tilde{S}_f(n, k) = 0$. Note that $\tilde{S}_f(n, 1) = S_f(n, 1) = 1$. The limit $q \to -1$ of the q-deformed commutation relations (24) does not imply the exclusion principle—this has to be added. To obtain a Fock space representation of (24), a basis $|n\rangle_q$ is constructed out of the vacuum $|0\rangle_q$ as follows [18, 19]

$$|n\rangle_q \equiv \frac{1}{\sqrt{[n]_q!}}\left(b_q^\dagger\right)^n |0\rangle_q.$$

Due to (3) one has to impose in the limit $q \to -1$ the exclusion principle in a weak sense, i.e., $(b^\dagger)^n|0\rangle_{q=-1} = 0$ for $n \geqslant 2$ (but recall that it is not true that $(b^\dagger)^n = 0$). This *weak exclusion principle* is discussed, e.g., in [18–21]. Thus, the coefficients $S_f(n, k)$ (obtained from a limit $q \to -1$) contain 'more information' than the coefficients $\tilde{S}_f(n, k)$ of the limit case $q = -1$ where the exclusion principle holds true. The combinatorial structure of the fermionic case (i.e., in the limit case $q = -1$ together with the exclusion principle) was already discussed in [25] for the multimode case and it would be interesting to understand in detail how the structures discussed in [25] emerge in the limit $q \to -1$ from the q-deformed case, in particular their relation to the results discussed above. It was shown in [5] that (for $q > 0$) the q-deformed Stirling numbers $S_q(n, k)$ have an interpretation as expectation values of the operators $\left(b_q^\dagger b_q\right)^n$ with respect to a Fock space basis, and it was shown in [26] that the q-deformed Bell numbers $B_q(n)$ give the expectation value of $\left(b_q^\dagger b_q\right)^n$ with respect to coherent states (see also [15] for a discussion). It should be interesting to find out whether a similar interpretation can be given for $S_f(n, k)$ and $B_f(n)$ (due to the weak exclusion principle this seems to be nontrivial). From a purely mathematical point of view, it would be interesting to find a nice combinatorial interpretation for the fermionic Stirling numbers and in particular an interpretation for those combinations (n, k) where $S_f(n, k) < 0$. For first results see [16]. The q-deformed Stirling numbers have been discussed in a combinatorial context by several authors, see, e.g., [6, 10–14, 16]. However, it seems that taking the limit $q \to -1$ in these considerations is not possible or at least unnatural (in the literature cited above it is normally assumed that $q \geqslant 0$, [16] being an exception). Thus, one should look for an intrinsic interpretation respecting the fermionic nature of the situation.

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